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Gauge equivalence of representations of symmetry groups in quantum mechanics

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Abstract. The equivalence of representations of symmetry groups operating upon wavefunctions in configuration space is studied with regard to the (intuitive) notion of physical equivalence. A refinement of the usual projective equivalence relation is introduced, called gauge equivalence, for which the allowed unitary equivalence transformations are gauge transformations. For a Euclidean as well as for a Newton–Hooke symmetry group the gauge equivalence classes of unitary multiplier representations are determined. These examples support the assertion that equivalence from a physical viewpoint corresponds better to this new gauge equivalence concept than to the usual notion of projective equivalence.

1. Introduction

Representations of symmetry groups play an important role in quantum mechanics, especially the *unitary multiplier representations*. Such a representation U obeys a relation of the form

$$U(g')U(g)=\mu(g',g)U(g'g) \tag{1}$$

where $\mu(g',g)$ is a phase factor (convention: $U(e)=\mathbf{1}$). Usually one considers two such unitary multiplier representations U and U' of a group G as equivalent if they are connected by the *projective equivalence relation*

$$U'(g)=\nu(g)TU(g)T^{-1} \tag{2}$$

where $\nu(g)$ is a phase factor and T a unitary transformation.

The freedom of an arbitrary phase factor $\nu(g)$ in the projective equivalence relation originates from the interpretation of rays rather than vectors as physical states, which is a well founded principle in quantum mechanics. On the other hand, the freedom of an arbitrary unitary equivalence transformation T has a physically much weaker foundation. Such a transformation conserves the structure of an *abstract* Hilbert space, i.e. the structure common to all Hilbert spaces, especially the inner product. However, the *concrete* Hilbert spaces in quantum mechanics have more structure than just that. For systems described by wavefunctions in space(–time) not only the inner product, integrated over the whole space, but also the local 'inner product density' has a physical meaning, related to probability density. In general, the

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information about these *local* quantities will be destroyed by a *global* unitary equivalence transformation. Hence, as Bargman and Wigner (1948) have already noticed, 'it cannot be claimed that equivalence in the sense of (2) implies equivalence in every physical aspect'. This becomes obvious when one realises with Dirac (1962) that the equivalence relation (2) 'involves looking upon all unitary transformations as trivial'. A physicist will consider a transformation as trivial only if it conserves all physical properties in which he is interested, and this strongly depends on the physical context in which he applies that transformation. In this paper the physical context will be that of 'elementary' quantum mechanics described by wavefunctions in configuration space. Therefore it is natural here to consider a unitary transformation as trivial if it is a gauge transformation leaving not only the global but also the local properties invariant.

For that reason we will define in this paper another equivalence concept for unitary multiplier representations in configuration space, called *gauge equivalence.* This will be a refinement of projective equivalence by the restriction that the unitary transformations T allowed in equation (2) have to be gauge transformations. The point is that 'equivalence in every physical aspect' corresponds better to this new gauge equivalence concept than to the usual notion of projective equivalence. We will come back to this assertion in the conclusion.

In a previous paper (Hoogland 1976) we have already started an investigation of the discrepancy between projective and 'physical' equivalence, especially with regard to the notion of superequivalence of group exponents (Lévy-Leblond 1969). That discrepancy, however, will also be present in cases of symmetry groups for which superequivalence does not play a role. In our present approach we can deal with the problem, irrespective as to whether superequivalence of exponents is a relevant notion for the group at issue. (In fact, gauge equivalence will be a refinement not only of projective equivalence but also of the so-called 'local equivalence' introduced on p 435 of Hoogland (1976).)

This paper will be organised as follows. In § 2 the gauge equivalence relation for unitary multiplier representations in configuration space is introduced. In § 3 some technical details are given, necessary for the formulation of a result about a complete set of inequivalent gauge functions. In the §§ 4 and 5 two examples are given: for a Euclidean and for a Newton–Hooke symmetry group the projective (§§ 4.1 and 5.1) as well as the gauge (§§ 4.2 and 5.2) equivalence classes of unitary multiplier representations are determined and a physical interpretation (§§ 4.3 and 5.3) is given. A conclusion is given in the last section.

2. The main concept

Let a group G and a set X be given, with a transitive operation of G upon X. We will consider the unitary multiplier representations $U(g)$ operating in Hilbert space $\mathscr{H}$ spanned by complex functions $\psi(x)$ such that

$$|(U(g)\psi)(gx)|^2 = |\psi(x)|^2 \tag{3}$$

or, in other words, such that $|\psi(x)|^2$ transforms as a scalar function. Throughout this section we will assume these concrete properties added to the abstract structure of the group G, the Hilbert space $\mathscr{H}$ and the representation U.

The set X may be thought of as a certain space(–time) and the functions $\psi(x)$ as wavefunctions, so that $|\psi(x)|^2$ may be interpreted as a probability density. However, in this section we will not use any topological property of G and X and, accordingly, we will not specify an explicit form for the inner product and the equation of motion of the wavefunctions.

The concrete Hilbert space structure introduced above will be sufficient in order to use the notion of gauge transformation between two Hilbert spaces, by which we mean a unitary transformation $T: \mathcal{H} \to \mathcal{H}'$ with the property

$$|(T\psi)(x)|^2 = |\psi(x)|^2. \tag{4}$$

By means of these gauge transformations we can define the main concept of this paper:

Definition 1. Two unitary multiplier representations U and U' of G in $\mathcal{H}$ and $\mathcal{H}'$ are called *gauge equivalent* iff there exist a phase function $\nu(g)$ and a gauge transformation $T: \mathcal{H} \to \mathcal{H}'$ such that equation (2) holds.

This definition obviously gives a refinement of the usual projective equivalence relation. The physical idea behind this new equivalence concept has been given in the introduction.

Equation (3) implies the existence of a phase function $A(g, x)$ such that the unitary $U(g)$ operates as

$$(U(g)\psi)(gx) = A(g, x)\psi(x). \tag{5}$$

The function $A(g, x)$ is called a *gauge function.* From (1) it follows that such a gauge function has the property

$$A(g', gx)A(g, x) = \mu(g', g)A(g'g, x) \tag{6}$$

with the convention $A(e, x) = 1$. Equation (4) implies the existence of a phase function $S(x)$ such that the unitary T operates as

$$(T\psi)(x) = S(x)\psi(x). \tag{7}$$

The concept of gauge equivalence of representations 'induces' an equivalence relation for the gauge functions:

Definition 2. Two gauge functions A and A' are called *equivalent* iff there exist phase functions $\nu(g)$ and $S(x)$ such that

$$A'(g, x) = \nu(g)S(gx)A(g, x)/S(x). \tag{8}$$

It is obvious that two gauge equivalent representations U and U' have equivalent gauge functions A and A'. The inverse is not true: two representations having equivalent gauge functions are not necessarily gauge equivalent, not even when they are projectively equivalent. The reason is that the function $S(x)$ obeying equation (8) does not necessarily define a unitary transformation between the two Hilbert spaces that carry the representations.

Theorem 1. Two unitary multiplier representations in $\mathcal{H}$ and $\mathcal{H}'$ are gauge equivalent iff their gauge functions are equivalent *and* equation (8) can be satisfied by a phase function $S(x)$ defining a unitary transformation $T: \mathcal{H} \to \mathcal{H}'$ by equation (7).

The proof of this statement is easy enough to leave it as an exercise for the reader.

A generalisation of the foregoing concepts to unitary/anti-unitary multiplier representations operating on wavefunctions with several components can be found in part I of Hoogland (1977).

3. Technical considerations

From theorem 1 it follows that a complete set of inequivalent gauge functions is a necessary (but not sufficient) ingredient for the determination of the different gauge equivalence classes of unitary multiplier representations. The technique of obtaining such a complete set of gauge functions has been developed by Lévy-Leblond (1969) for the Lagrangian gauge problem in classical mechanics. This technique has been extended for application in quantum mechanics (Hoogland 1976, 1977). Here we only give those details that are necessary for a formulation of the result.

The fact that G operates *transitively* upon X means that any given point x_0 of X can be mapped onto all other points x of X by the operation of group elements h_x of G, so that $h_x x_0 = x$. Let x_0 and $\{h_x\}_{x\in X}$ be chosen fixed for once and all (convention: $h_{x_0} = e$). Let Γ be the subgroup of G that leaves x_0 invariant. The elements h_x are representatives for the left co-sets of Γ in G. There is a unique decomposition $g = h(g)\gamma(g)$ with $h(g) = h_{gx_0}$ and $\gamma(g) \in \Gamma$.

Let μ be a multiplier of G (Parthasarathy 1969) with the extra property (called *centralisation*)

$$\mu(g, \gamma) = 1 \qquad (\forall g \in G, \forall \gamma \in \Gamma). \tag{9}$$

Let ρ be a one-dimensional unitary representation (*character*) of Γ. Then A defined by

$$A(g, x) = \mu(g, h_x)/\rho(\gamma(gh_x)) \tag{10}$$

is a gauge function, and μ is 'the' multiplier of A, i.e. A and μ obey equation (6). Any equivalence class of multipliers of G that are trivial on Γ contains a centralised representative obeying equation (9), and a collection of such representatives will be called a complete set of centralised multipliers of G.

The characters of Γ form an Abelian group under pointwise multiplication, and those characters of Γ that can be extended to a character of G form a subgroup. A collection of representatives from the co-sets of this subgroup will be called a complete set of non-extensible characters of Γ.

Theorem 2. If μ moves over a complete set of centralised multipliers of G and if ρ moves over a complete set of non-extensible characters of Γ then A defined by equation (10) moves over a complete set of inequivalent gauge functions.

The proof of this result (in different formulations) can be found in the literature. In particular, equation (10) of this paper corresponds to equation (43) of Lévy-Leblond (1969) and equation (34) of Hoogland (1976).

For the classification of unitary multiplier representations, up to projective as well as up to gauge equivalence, we will need the ordinary (non-projective) equivalence concept, which we will emphatically call *unitary equivalence* in the following sections. So U and U' are unitarily equivalent if they are projectively equivalent in such a way that equation (2) holds for $\nu(g) \equiv 1$.

4. Application to a Euclidean group

Let G be the covering group of the Euclidean transformations in a plane, obtained by replacement of the rotation subgroup SO(2) by its covering group. The group elements of G will be denoted by $g=(\boldsymbol{a}, \phi)$; we do not consider inversions. The operation of G upon the x–y plane and the group product in G are as usual:

$$g\boldsymbol{x} = R(\phi)\boldsymbol{x} + \boldsymbol{a} \tag{11}$$

$$g'g = (\boldsymbol{a}' + R(\phi')\boldsymbol{a}, \phi' + \phi). \tag{12}$$

We want to classify the multiplier representations of G, first up to projective equivalence (see § 4.1) and then up to gauge equivalence (see § 4.2). The assertion that gauge equivalence rather than projective equivalence corresponds to 'equivalence in every physical aspect' will be confirmed by the physical interpretation (see § 4.3).

4.1. Classification up to projective equivalence

The multipliers of G can be given, up to equivalence, in the following form, where the label β moves over the real numbers (see section 6d III, p 37, Bargmann 1954):

$$\mu_\beta(\boldsymbol{a}', \phi'; \boldsymbol{a}, \phi) = \exp[\tfrac{1}{2}\mathrm{i}\beta(\boldsymbol{a}' \times R(\phi')\boldsymbol{a})_z]. \tag{13}$$

If the unitary multiplier representation U with multiplier μ_β is parametrised as

$$U(\boldsymbol{a}, \phi) = \exp(-\mathrm{i}\boldsymbol{a}.\boldsymbol{P})\exp(-\mathrm{i}\phi J) \tag{14}$$

then the commutation relations of the infinitesimal (Hermitian) generators are

$$[P_x, P_y] = -\mathrm{i}\beta, \qquad [J, P_x] = \mathrm{i}P_y, \qquad [J, P_y] = -\mathrm{i}P_x. \tag{15}$$

The operator C defined by

$$C = \boldsymbol{P}^2 - 2\beta J \tag{16}$$

commutes with all generators and, hence, in an irreducible representation C is a (real) multiple of the unit operator. The irreducible unitary multiplier representations with non-trivial multiplier are labelled (up to *unitary* equivalence) by $\beta \in \mathbb{R} - \{0\}$ and $C \in \mathbb{R}$.

A standard realisation, denoted by $U_{[\beta,C]}$, operating in the Hilbert space of square integrable functions $\phi(k)$ is given by (14) with

$$P_x = k, \qquad P_y = \mathrm{i}\beta\frac{\mathrm{d}}{\mathrm{d}k}, \qquad J = \frac{1}{2\beta}\left(k^2 - \beta^2\frac{\mathrm{d}^2}{\mathrm{d}k^2} - C\right). \tag{17}$$

From (14) and (17) it is easily seen that

$$U_{[\beta,C']}(\boldsymbol{a}, \phi) = \exp[\mathrm{i}\phi(C' - C)/2\beta]U_{[\beta,C]}(\boldsymbol{a}, \phi) \tag{18}$$

which means that $U_{[\beta,C']}$ and $U_{[\beta,C]}$ are *projectively* equivalent. This completes the classification of the unitary multiplier representations of the present group G, up to unitary as well as up to projective equivalence.

4.2. Classification up to gauge equivalence

So far we only used well known representation theoretical methods applied to the abstract group structure of G. Here we will also use the concrete structure provided by the operation of G upon the plane, and we will apply the results of §§ 2 and 3.

The obvious choice for x_0 will be the origin $\mathbf{0}$ in the plane. Then its invariance group Γ is the covering group of the rotations. The co-set representatives h_x will be chosen simply as the translations from $\mathbf{0}$ to $\boldsymbol{x}$. So in this case the set of elements $\{h_{\boldsymbol{x}}\}$ forms a subgroup, viz. the translation subgroup. This is even a normal subgroup and the structure of G is that of a semi-direct product of the translation subgroup with Γ. Due to this structure all characters of Γ can be trivially extended to a character of G. Substitution of $\boldsymbol{a}=\mathbf{0}$ in (13) shows that all multipliers μ_β are centralised (see equation (9)).

From theorem 2 it now follows that a complete set of inequivalent gauge functions is given by

$$A_\beta(\boldsymbol{a},\phi;\boldsymbol{x})=\exp[\tfrac{1}{2}\mathrm{i}\beta(\boldsymbol{a}\times R(\phi)\boldsymbol{x})_z]. \tag{19}$$

Substitution of this result in equation (5) gives the form of the operators $U(\boldsymbol{a},\phi)$ in a Hilbert space of wavefunctions $\psi(\boldsymbol{x})$. By differentiation we obtain the infinitesimal generators operating in configuration space:

$$P_x=-\mathrm{i}\,\partial_x-\tfrac{1}{2}\beta y, \qquad P_y=-\mathrm{i}\,\partial_y+\tfrac{1}{2}\beta x \qquad J=-\mathrm{i}(x\,\partial_y-y\partial_x). \tag{20}$$

We want to know which of the irreducible unitary multiplier representations $U_{[\beta,C]}$, given by (14) and (17), are unitarily equivalent to a representation of the form, given by (14) and (20). To this end we look for unitary transformations $\phi(k)\to\psi(\boldsymbol{x})$ transforming (17) into (20). The most general *linear* transformation which does the trick has the form

$$\psi(\boldsymbol{x})=\int_{-\infty}^{+\infty}\frac{\mathrm{d}k}{2\pi}F(k)\exp\left(\mathrm{i}xk-y\beta\frac{\mathrm{d}}{\mathrm{d}k}\right)\phi(k) \tag{21}$$

where the function $F(k)$ is a solution of

$$\left(k^2-\beta^2\frac{\mathrm{d}^2}{\mathrm{d}k^2}-C\right)F(k)=0. \tag{22}$$

If we define the inner product for the wavefunctions as usual then *unitarity* means

$$\int_{-\infty}^{+\infty}\mathrm{d}\boldsymbol{x}\psi^*(\boldsymbol{x})\psi(\boldsymbol{x})=\int_{-\infty}^{+\infty}\frac{\mathrm{d}k}{2\pi}\phi^*(k)\phi(k). \tag{23}$$

This results in a square integrability condition on $F(k)$

$$\int_{-\infty}^{+\infty}\mathrm{d}k|F(k)|^2=|\beta|. \tag{24}$$

The equations (22) and (24) together are well known, e.g. from the eigenvalue problem of the harmonic oscillator. Only for a discrete spectrum of eigenvalues

$$C=(2n+1)|\beta|, \qquad n=0,1,2,\ldots \tag{25}$$

equation (22) has square integrable solutions, viz. Hermite functions

$$F(k)=|\beta|^{1/4}h_n(k/|\beta|^{1/2}). \tag{26}$$

The result so far is the following: only if C belongs to the discrete spectrum (25) then $U_{[\beta,C]}$ is unitarily equivalent to a representation of the form determined by (20), operating on wavefunctions $\psi(\boldsymbol{x})$. We will denote these representations by $U_{\{\beta,n\}}$. Their carrier space is the Hilbert space spanned by the functions $\psi(\boldsymbol{x})$ given by (21) and (26).

It will be clear that $U_{\{\beta,n'\}}$ and $U_{\{\beta,n\}}$ are still projectively equivalent because $U_{[\beta,C']}$ and $U_{[\beta,C]}$ are so. However $U_{\{\beta,n'\}}$ and $U_{\{\beta,n\}}$ are not gauge equivalent because a gauge transformation between their carrier Hilbert spaces does not exist. This can be seen most easily from the equation obtained from (16) by substitution of (20) and (25):

$$(-\mathrm{i}\,\partial_x+\tfrac{1}{2}\beta y)^2+(-\mathrm{i}\,\partial_y-\tfrac{1}{2}\beta x)^2=(2n+1)|\beta|. \tag{27}$$

Wavefunctions obeying this equation for different values of n are certainly not related by a gauge transformation.

Herewith we have completed the classification, up to gauge equivalence, of the unitary multiplier representations (with non-trivial multiplier). A set of representatives is given by the $U_{\{\beta,n\}}$ with $\beta\in\mathbb{R}-\{0\}$ and $n=0,1,2,\ldots$.

4.3. Physical interpretation

The present group G and its representations are very useful for the description of a charged particle in a uniform magnetic field, relativistic as well as non-relativistic. The connected invariance group of a uniform magnetic field $\boldsymbol{B}$ is the direct product of the Euclidean group in a plane perpendicular to $\boldsymbol{B}$ and of the Galilei (or Poincaré) group along a line parallel to $\boldsymbol{B}$. Moreover, it can be shown that the unitary multiplier representations of that invariance group are the tensor products of such representations of its factors (Hoogland 1977, 1978). If we disregard the free motion of the particle parallel to $\boldsymbol{B}$ then we can restrict ourselves to the Euclidean group and its representations. If we write $\beta=eB$ and $C=2m\mathscr{E}$ where m and e are the mass and the charge of the particle, then we obtain from (16)

$$\mathscr{E}=\frac{\boldsymbol{P}^2}{2m}-\frac{e}{m}BJ. \tag{28}$$

In this relation we recognise a kinetic energy term and a magnetic energy term. Apparently the constant $\mathscr{E}$ is to be identified with the energy of the motion of the particle perpendicular to the field.

The results obtained under § 4.1 then mean that C and, hence, $\mathscr{E}$ may have *all* real values. Moreover, different values give rise to projectively *equivalent* representations. This would mean that $\mathscr{E}$ here would play a role similar to that of the internal energy $\mathscr{V}$ in the multiplier representations of the Galilei group (Lévy-Leblond 1974) and of the Newton–Hooke group (see the next section). However, we know that this is not true. The values of $\mathscr{E}$ characterise the discrete Landau levels which are not at all equivalent in every physical aspect. A transition between two such levels is not a trivial transformation of the mathematical description (like changing the zero point of energy) but it is a physical process involving emission or absorption of energy.

Fortunately, by the results obtained under § 4.2 these problems are cleared away. Indeed, equation (25) results in a *discrete* spectrum for the Landau energy

$$\mathscr{E}=(n+\tfrac{1}{2})\frac{|eB|}{m}, \qquad n=0,1,2,\ldots \tag{29}$$

and all different values of $\mathscr{E}$ give rise to gauge *inequivalent* representations.

Notice that equation (27) results in (the x–y part of) the wave equation in which the magnetic field $\boldsymbol{B}$ is involved by minimal coupling to its potential $\boldsymbol{A} = (-\frac{1}{2}By, \frac{1}{2}Bx, 0)$ in the so called symmetric gauge:

$$\frac{1}{2m}[(-\mathrm{i}\,\partial_x - eA_x)^2 + (-\mathrm{i}\,\partial_y - eA_y)^2] = \mathscr{E}. \tag{30}$$

For more details about the jion between minimal coupling and symmetry we refer to a previous paper (Hoogland 1978).

5. Application to a cosmological group

Let G be the group of transformations $g = (b, a, v)$ operating upon the events $x = (t, z)$ in a one-(space-)dimensional universe as follows:

$$gx = (t + b, z + v \sin t + a \cos t). \tag{31}$$

From the operation of g' upon the event gx one easily finds the group product in G:

$$g'g = (b' + b, a' \cos b + v' \sin b + a, v' \cos b - a' \sin b + v). \tag{32}$$

This group (or rather its three-dimensional analogue) has been introduced by Bacry and Lévy-Leblond (1968) as a contraction of a de Sitter group (see also Sudbery 1972). Bacry and Lévy-Leblond have called it a Newton group, and under this name it has been dealt with by Lévy-Leblond (1969) and Hoogland (1976). Derome and Dubois (1972) preferred to associate the name of Hooke to this group, and under that name it has been considered by Dubois (1973a, b) and by Roman and Haavisto (1976). Here we will call this group the Newton–Hooke group. We will classify the multiplier representations of this group G analogously to § 4.

5.1. Classification up to projective equivalence

The multipliers of G, up to equivalence, can be given in the following form, where the label λ moves over the real numbers (see formula (17*a*) of Derome and Dubois 1972):

$$\mu_\lambda(b', a', v'; b, a, v) = \exp[\mathrm{i}\lambda(\tfrac{1}{2}(v'^2 - a'^2) \cos b \sin b + v'a \cos b - a'a \sin b - v'a' \sin^2 b)]. \tag{33}$$

If the unitary multiplier representation U with multiplier μ_λ is parametrised as

$$U(b, a, v) = \exp(\mathrm{i}bH) \exp(-\mathrm{i}aP) \exp(\mathrm{i}vN) \tag{34}$$

then the commutation relations of the infinitesimal (Hermitian) generators are

$$[P, N] = -\mathrm{i}\lambda, \qquad [H, P] = \mathrm{i}N, \qquad [H, N] = -\mathrm{i}P. \tag{35}$$

The operator C defined by

$$C = P^2 + N^2 - 2\lambda H \tag{36}$$

commutes with all generators and, hence, in an irreducible representation C is a (real) multiple of the unit operator. The irreducible unitary multiplier representations with non-trivial multiplier are labelled (up to *unitary* equivalence) by $\lambda \in \mathbb{R} - \{0\}$ and $C \in \mathbb{R}$. A standard realisation, denoted by $U_{[\lambda,C]}$, operating in the Hilbert space of square integrable functions $\phi(p)$ is given by (34) with

$$P = p, \qquad N = \mathrm{i}\lambda \frac{\mathrm{d}}{\mathrm{d}p}, \qquad H = \frac{1}{2\lambda}\left(p^2 - \lambda^2 \frac{\mathrm{d}^2}{\mathrm{d}p^2} - C\right). \tag{37}$$

From (34) and (37) it is easily seen that

$$U_{[\lambda,C']}(b, a, v) = \exp[-\mathrm{i}b(C' - C)/2\lambda]U_{[\lambda,C]}(b, a, v) \tag{38}$$

which means that $U_{[\lambda,C']}$ and $U_{[\lambda,C]}$ are *projectively* equivalent.

5.2. *Classification up to gauge equivalence*

The obvious choice for x_0 will be the event (0, 0), i.e. the origin in the t–z plane. From (31) it is clear that the invariance group Γ of x_0 consists of all group elements of the form $(0, 0, v)$. The left co-set representatives $h_{(t,z)}$ transforming the origin (0, 0) into the event (t, z) will be chosen as the group elements of the form $(t, z, 0)$. The decomposition $g = h(g)\gamma(g)$ now reads $(b, a, v) = (b, a, 0)(0, 0, v)$. In the present case the set of elements $\{h_{(t,z)}\}$ does not form a subgroup, so it is hardly useful to call them 'translations'. The characters of Γ, given by $\rho_\alpha(v) = \exp(\mathrm{i}\alpha v)$ with $\alpha \in \mathbb{R}$, cannot be extended to characters of G, except the trivial one for which $\alpha = 0$. Substitution of $b = a = 0$ in (33) shows that all multipliers μ_λ are centralised.

From theorem 2 it now follows that a complete set of inequivalent gauge functions is given by

$$\begin{aligned} A_{\lambda,\alpha}(b, a, v; t, z) &\\ = \exp\{&\mathrm{i}\lambda\,[\tfrac{1}{2}(v^2 - a^2)\cos t \sin t + vz \cos t - az \sin t - va \sin^2 t] \\ &- \mathrm{i}\alpha(v \cos t - a \sin t)\}. \end{aligned} \tag{39}$$

Substitution of this result in equation (5) gives the form of the operators $U(b, a, v)$ in a Hilbert space of wavefunctions $\psi(t, z)$. By differentiation we obtain the infinitesimal generators operating in configuration space:

$$\begin{aligned} H &= \mathrm{i}\,\partial_t \\ P &= -\mathrm{i}(\cos t)\partial_z + (\lambda z - \alpha)\sin t \\ N &= \mathrm{i}(\sin t)\partial_z + (\lambda z - \alpha)\cos t. \end{aligned} \tag{40}$$

The most general *linear* transformation $\phi(p) \to \psi(t, z)$ transforming the generator H in (37) into H in (40) has the form

$$\psi(t, z) = \int_{-\infty}^{+\infty} \frac{\mathrm{d}p}{2\pi} F(p, z) \exp\left[\mathrm{d}-\mathrm{i}t\left(p^2 - \lambda^2 \frac{\mathrm{d}^2}{\mathrm{d}p^2} - C\right)(2\lambda)^{-1}\right]\phi(p) \tag{41}$$

where the function $F(p, z)$ has to be determined from the condition that also the generators P and N in (37) transform into P and N in (40). It is a matter of straightforward calculation to find that this condition gives the following equations:

$$(\mathrm{i}\,\partial_z + p)F(p, z) = 0, \qquad (\mathrm{i}\,\partial_p + z - \alpha/\lambda)F(p, z) = 0. \tag{42}$$

A solution of these equations is

$$F(p, z) = \exp[ip(z - \alpha/\lambda)]. \tag{43}$$

This solution is unique up to a complex constant factor which by *unitarity* of the transformation (41) has modulus one. By gauge equivalence it is allowed to change the phase of that factor, so we may, indeed, choose it equal to one.

The result so far is the following: each multiplier representation $U_{[\lambda,C]}$ is unitarily equivalent to a set (labelled by α) of representations of the form determined by (40), operating on wavefunctions $\psi(t, z)$. We will denote these representations by $U_{\{\lambda,\alpha,C\}}$. Their carrier space is the Hilbert space spanned by the functions $\psi(t, z)$ given by (41) and (43).

It will be clear that $U_{\{\lambda,\alpha,C\}}$ and $U_{\{\lambda,\alpha',C\}}$ are gauge *in*equivalent, as they have inequivalent gauge functions $A_{\lambda,\alpha}$ and $A_{\lambda,\alpha'}$. On the other hand, $U_{\{\lambda,\alpha,C\}}$ and $U_{\{\lambda,\alpha,C'\}}$ are gauge equivalent; calculation shows that

$$U_{\{\lambda,\alpha,C'\}}(b, a, v) = \exp[-ib(C' - C)/2\lambda]TU_{\{\lambda,\alpha,C\}}(b, a, v)T^{-1} \tag{44}$$

where T is the gauge transformation from the carrier space of $U_{\{\lambda,\alpha,C\}}$ onto that of $U_{\{\lambda,\alpha,C'\}}$ working on the wavefunctions as follows

$$(T\psi)(t, z) = \exp[it(C' - C)/2\lambda]\psi(t, z). \tag{45}$$

By substitution of (40) in (36) we obtain a differential equation for the functions $\psi(t, z)$:

$$\mathrm{i}\,\partial_t = \frac{1}{2\lambda}(-\mathrm{i}\,\partial_z)^2 + \frac{1}{2}\lambda z^2 - \alpha z + \frac{\alpha^2 - C}{2\lambda}. \tag{46}$$

It is easily checked that the gauge transformation T transforms the wavefunctions ψ obeying equation (46) into wavefunctions $\psi' = T\psi$ obeying an equation analogous to (46) where C is replaced by C'. Due to the gauge equivalence of $U_{\{\lambda,\alpha,C\}}$ and $U_{\{\lambda,\alpha,C'\}}$ we may choose C arbitrary for a representative from each gauge equivalence class. We will choose $C = \alpha^2$.

Herewith we have completed the classification, up to gauge equivalence, of the unitary multiplier representations with non-trivial multiplier. A set of representatives is given by the $U_{\{\lambda,\alpha,C\}}$ with $\lambda \in \mathbb{R} - \{0\}$, $\alpha \in \mathbb{R}$ and $C = \alpha^2$.

5.3. Physical interpretation

The Newton–Hooke group G may be interpreted as the kinematical symmetry group of a non-relativistic oscillating universe (Bacry and Lévy-Leblond 1968) or as the dynamical symmetry group of a harmonic oscillator (Sudbery 1972). Indeed, if we put $\lambda = m$ and $C = -2m\mathcal{V}$ then the relation (36) can be written in the form

$$H = \frac{1}{2m}P^2 + \frac{m}{2}\left(\frac{N}{m}\right)^2 + \mathcal{V}. \tag{47}$$

This is easily recognised as the Hamiltonian of a linear harmonic oscillator with mass m and $\omega = 1$, where P is the momentum operator, N/m is the position operator (Derome and Dubois 1972, § 12) and $\mathcal{V}$ is an additional internal energy. The interpretation of these quantities is completely analogous to that in the case of the Galilei group (Lévy-Leblond 1974). Apparently a 'free' particle in the oscillating Newton–Hooke universe behaves like a particle 'attached to a spring' in the Galilean universe (Lévy-Leblond 1969). The result obtained in § 5.1, viz. that we may choose the constant C and, hence, the internal energy $\mathscr{V}$ arbitrarily in a projective equivalence class, is in accordance with our non-relativistic interpretation.

However, the classification up to gauge equivalence gives more physically relevant information. The parameter α, labelling the different gauge equivalence classes that are contained within one projective equivalence class, has an obvious physical meaning. If we put $\alpha = f$ in (46) then we obtain with $\lambda = m$ the equation:

$$\mathrm{i}\,\partial_t = \frac{1}{2m}(-\mathrm{i}\,\partial_z)^2 + \frac{mz^2}{2} - fz. \tag{48}$$

Here we have chosen the internal energy so that the last term in (46) vanishes, as is allowed by gauge equivalence. Equation (48) is the Schrödinger equation of a harmonic oscillator (with $\omega = 1$) on which a constant external force f works, e.g. a uniform electric field. For different values of the force f the multiplier representations $U_{\{\lambda=m,\alpha=f,C\}}$ obtained under § 5.2 are projectively equivalent but not gauge equivalent. The latter concept then corresponds to the general idea of 'equivalence in every physical aspect', as a free particle ($f = 0$) is in some physical aspect inequivalent to a particle in an external field ($f \neq 0$), even in an oscillating universe.

6. Conclusion

Although one may not notice it at first sight, the examples in the previous two sections are isomorphic as abstract groups (exercise: find an explicit form of this isomorphism). Their operation upon space(–time), however, is not at all conserved by the isomorphism. In particular, the homogeneous subgroup Γ in § 4 will not be mapped by that isomorphism onto its analogue Γ in § 5, and the centralised multiplier (13) will not be mapped onto the centralised multiplier (33) but onto a non-centralised multiplier equivalent to (33). For that reason we did not exploit that group isomorphism.

Nevertheless, the very existence of that isomorphism will help to make our point clear. It shows that one cannot expect to obtain physically satisfactory results from representation theory by the use of the abstract group structure only, because then one obtains 'isomorphic' results for isomorphic groups (compare §§ 4.1 and 5.1). As isomorphic groups may be completely different in their operation upon space(–time), one has to use in some way the extra concrete structure provided by that operation. We have done so by choosing concrete forms for the Hilbert spaces $\mathscr{H}$ and the representations U that we considered (see equation (3)), and also for the equivalence transformations T that we allowed (see equation (4)). These forms are the natural ones within the framework of quantum mechanics, where the wavefunction $\psi(x)$ is interpreted as a probability amplitude density. As a result, gauge equivalence is the natural equivalence concept from a quantum mechanical point of view.

The examples in the previous two sections show that the classification up to gauge equivalence of the unitary multiplier representations corresponds well to the (admittedly intuitive) notion of 'equivalence in every physical aspect', whereas on the other hand the classification up to projective equivalence gives rise to an unsatisfactory or an incomplete physical interpretation.

The notion of 'local equivalence' introduced in a previous paper (Hoogland 1976) coincides with gauge equivalence for the Newton–Hooke group in § 5, and with projective equivalence for the Euclidean group in § 4. This is related to the fact that for the Newton–Hooke group superequivalence of exponents is a non-trivial concept, whereas it is trivial for the Euclidean group. So 'local equivalence' is intermediate between projective and gauge equivalence and only the latter concept gives satisfactory results for both examples. This justifies the introduction of the new gauge equivalence relation.

This paper can be generalised in two respects. First, one may consider representations consisting not only of unitary but also of anti-unitary operators; secondly, the representations may operate on wavefunctions having more than one component (Hoogland 1977). The present simplified account, however, is sufficient in order to show that the notion of gauge equivalence is relevant in quantum mechanics.

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